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Asymptotic analysis of extinction behaviour in fast nonlinear diffusion

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Abstract A number of initial-boundary-value problems for the equation of fast diffusion are analysed (at varying levels of detail and completeness), i.e.,

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^{-n} \nabla u)$$

with $n > 0$, in dimension $N > 2$ and with zero-Dirichlet boundary data, namely (i) the Cauchy problem (no boundary), mainly summarising existing results, (ii) the interior problem for a simply connected bounded domain (in large part revisiting earlier results), (iii) the problem exterior to a simply connected bounded domain and (iv) the half-space problem (for which we include $N = 2$). The critical (borderline) case $n = n_s \equiv 4/(N + 2)$, which arises in Yamabe flow, is the subject of particular focus, in part because it provides considerable insight into both the subcritical case, $0 < n < n_s$, and the supercritical one, $n_s < n < 1$. The results are based on formal-asymptotic analysis and suggest a range of conjectures that could be the subject of rigorous studies. The role of distinct types of similarity solutions is highlighted.

Keywords Coagulation-fragmentation · Extinction behaviour · Fast nonlinear diffusion · Self-similarity · Spike dynamics · Thin-film flows · Yamabe flows

1 Introduction

The equation of fast nonlinear diffusion,

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^{-n} \nabla u), \tag{1}$$

arises in a range of applications (and see [1,2], for example, for an account of the considerable progress that has been in the mathematical analysis of models of this class), including those we now briefly summarise.

(i) Flow of a very thin film of viscous liquid over a planar substrate driven by van der Waals forces (see, for example, [3]), where $u(\boldsymbol{x}, t)$ is the thickness of the fluid film. Here the driving force for spreading is in effect provided by a reduction in energy resulting in pertinent physical circumstances from the replacement

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of the solid–gas interface (associated with the substrate over which the fluid is able to spread being dry) by solid–liquid and liquid–gas ones. When this exchange is energetically favourable, the driving force associated with covering a dry substrate by a film that is sufficiently thin that van der Waals forces dominate (and thereby introducing both solid–liquid and liquid–gas interfaces) can lead, in a continuum framework, to a singularity in the effective diffusivity as $u \to 0$ such as that in (1).

(ii) Coagulation of diffusing particles. Consider particles that diffuse while coagulating and fragmenting, governed by the diffusive coagulation-fragmentation equations

$$\frac{\partial c_1}{\partial t} = D_1 \nabla^2 c_1 - \sum_{k=1}^{\infty} (a_{k,1} c_k c_1 - b_{k,1} c_{k+1}),$$
$$\frac{\partial c_j}{\partial t} = D_j \nabla^2 c_j + \frac{1}{2} \sum_{k=1}^{j-1} (a_{k,j-k} c_k c_{j-k} - b_{k,j-k} c_j) - \sum_{k=1}^{\infty} (a_{k,j} c_k c_j - b_{k,j} c_{k+j}), \tag{2}$$

where $j = 2, 3, \ldots$ labels the number of monomers in a given cluster, $a_{j,k} = a_{k,j}$ determines the rate of coagulation of clusters of size j and k into one of size $j + k$ and $b_{j,k} = b_{k,j}$ gives the rate of fragmentation of a cluster of size $j + k$ into one of size j and one of size k (the factor $1/2$ in the second of (2) avoids double counting of the relevant processes). The associated reactions thus take the form

$$P_j + P_k \underset{b_{j,k}}{\overset{a_{j,k}}{\rightleftharpoons}} P_{j+k},$$

where P_i denotes an oligomer made up of i monomers. The constants D_j are the diffusion coefficients, which can be assumed to decrease with increasing j.

We now consider the limit of fast coagulation and fragmentation (cf. the rigorous results in [4]; a very special case of the modelling approach that we outline is given in [5]): we thus assume quasi-equilibrium between coagulation and fragmentation, adopting detailed-balance assumptions to give

$$b_{k,j} c_{k+j} = a_{k,j} c_k c_j, \tag{3}$$

whereby

$$c_j = Q_j c_1^j, \quad Q_1 = 1, \tag{4}$$

where the Q_j are independent of $c_1(\boldsymbol{x}, t)$ and the rate coefficients are subject to the constraint that

$$b_{k,j}/a_{k,j} = Q_k Q_j / Q_{j+k}$$

for each k, j. The system (3), which pertains in the limit $a_{k,j}, b_{k,j} \to \infty$ with $a_{k,j}/b_{k,j} = \mathrm{O}(1)$, must be supplemented by the overall mass-conservation equation

$$\frac{\partial \rho}{\partial t} = \nabla^2 \left(\sum_{j=1}^{\infty} j D_j c_j \right), \quad \rho \equiv \sum_{j=1}^{\infty} j c_j, \tag{5}$$

which follows from (2) under suitable assumptions on the behaviour of the rate constants as $k, j \to \infty$. Given (3)–(4), so that

$$\rho(c_1) = \sum_{j=1}^{\infty} j Q_j c_1^j, \quad \Psi(c_1) = \sum_{j=1}^{\infty} j D_j Q_j c_1^j,$$

Eq. (5) can be viewed as a nonlinear diffusion equation for the total density ρ in which the effective diffusivity

$$D(\rho) = \frac{\mathrm{d}\Psi}{\mathrm{d}c_1} \Big/ \frac{\mathrm{d}\rho}{\mathrm{d}c_1}$$

is typically a decreasing function of ρ (coagulation becomes more pronounced as the density increases, leading to clusters of larger size and hence to a reduction in the rate of transport). Two special cases serve to demonstrate the scope for relevance of the power-law case (1) to such applications. Firstly, if most of the mass is contained in clusters of size $j = J$, but transport is dominated by monomers (i.e., $Q_J \gg Q_j$ for $j \neq J$ but $D_1 Q_1 \gg D_j Q_j$ for $j \neq 1$), then

$$D(\rho) \sim \hat{D}\rho^{-n}, \quad n \equiv 1 - 1/J, \quad \hat{D} \equiv D_1/J(JQ_J)^{1/J}, \tag{6}$$

the upper bound of $n = 1$ implied by (6) being noteworthy. Secondly, the case

$$Q_j = (c^*)^{-(j-1)}, \quad \rho = \frac{c_1}{(1 - c_1/c^*)^2}$$

with constant c^* is also instructive (note that bounded density requires $c_1 < c^*$). Taking, purely for illustrative purposes, $D_j = \hat{D}/j$ then implies

$$\Psi(c_1) = \hat{D}\frac{c_1}{1 - c_1/c^*}, \quad D(\rho) = \hat{D}\frac{1 - c_1/c^*}{1 + c_1/c^*},$$

so that $D(\rho) \propto \rho^{-1/2}$ as $c_1 \to c^*$, corresponding to power-law behaviour at high densities.

(iii) Population spreading with mating pressure. Equation (1) with $u > 0$ provides a simple description of a population (density u) whose rate of spread increases at low densities, a phenomenon that is sometimes associated with the decreased probability of finding a mate in a given region leading to an increase in the dispersal rate (see [6] and references therein); the converse case $n < 0$, associated with an increase in the diffusivity at high densities, has been used to describe the effects on dispersal of overcrowding [7] (i.e., driven by 'population pressure' rather than 'mating pressure').

The remainder of the paper is organised as follows. Section 2 introduces the types of similarity solutions with which we shall be concerned. Section 3 revisits known symmetry results, both for (1) and for the elliptic problem associated with the reduction (13), that will be useful in the subsequent sections. Section 4 sets up some integral properties that are also needed later and applies a special case of these in summarising existing results for the Cauchy problem. Section 5 similarly outlines established behaviour for the interior problem, but also (in conjunction with Appendix 1) analyses the non-radially symmetric version of the critical case $n = n_s$. Section 6 concerns the exterior problem, for which results in the fast diffusion have previously been restricted to the subcritical case $n < n_c$. Section 7 investigates the dipole problem, identifying a new critical exponent $n = n_d$ and providing a comprehensive classification of the intermediate-asymptotic behaviour for the fast-diffusion case $n > 0$ (previous analyses having focussed on slow diffusion, $n < 0$). Appendix 2 discusses the 'spike' (i.e., $n \to n_s^-$) limit of the dipole problem, in which the profile for u is concentrated about a specific location $\boldsymbol{x}_c(\tau)$; this limit is particularly amenable to analytical study, and we accordingly also exploit related results elsewhere (notably in Appendix 1). Appendix 2 also contains the only detailed treatment here of a two-dimensional problem, namely the dipole one in the limit $n \to 1^-$. A summary of the parameter regimes that arise in these sections is given in Table 1. The discussion in Section 7 focusses on the status of the critical case $n = n_s$ and highlights formal results that could provide worthy topics for rigorous study.

2 Similarity solutions and intermediate-asymptotic behaviour

Equations such as (1), having two rescaling symmetries, possess the classes of similarity reduction

$$u = t^{-\alpha} f(\boldsymbol{\eta}), \quad \boldsymbol{\eta} = \boldsymbol{x}/t^{\beta}, \tag{7}$$

$$u = (t_c - t)^a f(\boldsymbol{\eta}), \quad \boldsymbol{\eta} = (\boldsymbol{x} - \boldsymbol{x}_c)/(t_c - t)^b, \tag{8}$$

where the extinction time t_c and location $\boldsymbol{x}_c$ in (8) typically depend on the initial data[1] and where the PDE provides only one constraint on (α, β) or (a, b) namely

$$\beta = (1 + n\alpha)/2, \quad b = (1 - na)/2 \tag{9}$$

[1] We do not include in (7) the degrees of freedom associated with translations in $\boldsymbol{x}$ and in t because (7) applies as $t \to +\infty$ and in this limit their incorporation leads to contributions of size $t^{-(\alpha+\beta)}$ and $t^{-(\alpha+1)}$, respectively, which (because $\beta > 0$) are smaller than the leading-order solution; by contrast, (8) holds as $t \to t_c^-$, so t_c must be included of necessity and $\boldsymbol{x}_c$ must also be when $b > 0$. It *is*, however, useful to include them in (7) when seeking refined large-time asymptotics (cf. [8], for example).

Table 1 Summary of the generic behaviour in the various regimes

Cauchy problem	$n < n_c$ Mass preserving (7), (18), (41)	$n_c < n < n_s$ Expanding second-kind (8), $b < 0$	$n_s < n < 1$ Contracting second-kind (8), $b > 0$	$1 < n$ No solution	Section 4
Interior problem	$n < n_s$ Separable (13)		$n_s < n < 1$ Contracting second-kind (8), $b > 0$	$1 < n$ No solution	Section 5
Exterior problem	$n < n_c$ Mass preserving as $t \to +\infty$ (7), (18), (41)	$n_c < n < n_s$ Expanding second-kind (8), $b < 0$	$n_s < n < 1$ Contracting second-kind (8), $b > 0$	$1 < n$ No solution	Section 6
Dipole problem	$n < n_d$ First-moment preserving (7), (57)	$n_d < n < n_s$ Expanding second-kind (8), $b < 0$	$n_s < n < 1$ Contracting second-kind (8), $b > 0$	$1 < n$ No solution	Section 7

The critical exponents, in addition to $n = 1$, are given by $n_c \equiv 2/N$, $n_s \equiv 4/(N+2)$, $n_d \equiv 2/(N+1)$. We note in the third column, the solution in the fourth row differs from that in the other two, in particular in not being radially symmetric

in the case of (1); we have chosen the signs in (7)–(8) so that we have $\alpha > 0$ (infinite-time extinction) or $a > 0$ (finite-time extinction) in all the phenomena we wish to explore ($\alpha > 0$ implies $\beta > 0$ for $n > 0$, but b can take either sign; t_c is a positive constant). Mechanisms by which the second constraint on the similarity exponents is determined (thereby completely specifying the required similarity reduction) have been discussed in detail, in [9] for example, in terms of first- and second-kind similarity solutions. However, we shall need here to add to this classification; before doing so, we note that the requirement that there be two scaling (say) invariants is only implicit in much of the discussion of first- and second-kind solutions—indeed, for equations with only one scaling symmetry, such as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^2, \tag{10}$$

the form of the scaling reduction, namely

$$u = f(x/t^{\frac{1}{2}})/t \tag{11}$$

in the case of (10), is of course determined completely without any reference to boundary conditions (and (11) indeed provides the large-time (intermediate-asymptotic) behaviour of the Cauchy problem for (10)). For the double-scaling-invariant problems we consider in this paper, we need to characterise three types of similarity solution, and we now outline this classification, whereby in the current context the relevant intermediate-asymptotic behaviour takes the form

$$u \sim t^{-\alpha} f(\boldsymbol{x}/t^{\beta}) \quad \text{as } t \to +\infty$$

in cases of infinite-time extinction (cf. (7)) and

$$u \sim (t_c - t)^a f((\boldsymbol{x} - \boldsymbol{x}_c)/(t_c - t)^b) \text{ as } \quad t \to t_c^-$$

(cf. (8)) when finite-time extinction occurs.[2]

[2] We shall in addition need to construct solutions that are not exact similarity reductions to (1), being logarithmically modified versions of (8) for example. Such logarithmic modifications have a long history in the study of blow-up problems in particular; see [10] for an early example.

(1) *Zeroth-kind self-similarity*
Here the second constraint on (α, β) or (a, b) is determined by requiring that the boundary conditions as well as the PDE be invariant under the scaling symmetries. Whenever the domain is bounded, this constraint suffices for our purposes: we shall consider (1) on either $\boldsymbol{x} \in \Omega$ or $\boldsymbol{x} \in \mathbb{R}^N \backslash \Omega$, where Ω is a bounded domain, with zero Dirichlet data

$$u = 0 \quad \text{for } \boldsymbol{x} \in \partial\Omega \tag{12}$$

on some fixed[3] (closed) surface $\partial\Omega$, requiring $b = 0$ in (8) and hence

$$u \sim (t_c - t)^{\frac{1}{n}} f(\boldsymbol{x}) \quad \text{as } t \to t_c^-. \tag{13}$$

Note that we shall never require the initial conditions to be invariant under the relevant symmetry: we seek solutions that describe possible intermediate-asymptotic behaviour for any initial data of a particular class. We also record here that (13) does not in fact describe this intermediate-asymptotic behaviour for all n, for reasons that we explore below.
While the imposition of the requirement that the boundary data be invariant under the relevant symmetry may be second nature to most workers on symmetry methods, discussions of similarity reductions for problems such as (1) have, as already noted, typically focussed on first- and second-kind reductions: it is rarely explicit in those discussions that the applicability of such reductions relies on the boundary data being invariant under both rescaling symmetries because the former apply at infinity, typically in the form

$$u \to 0 \quad \text{as } |\boldsymbol{x}| \to +\infty. \tag{14}$$

Indeed, under some circumstances a result such as (14) follows automatically from the decay at infinity of the initial data, so that *no* boundary condition need be imposed there and there is in consequence no such constraint to be imposed on the symmetry group.

(2) *First-kind self-similarity*
Here the extra condition on the similarity exponents is determined by an integral constraint, the most common one being conservation of mass whereby

$$M(t) \equiv \int\limits_{\mathbb{R}^N} u \, \mathrm{d}\boldsymbol{x} \tag{15}$$

is constant, which holds for the Cauchy problem for conservation laws such as (1) provided that the flux to infinity is zero, a proviso to which we shall return. Inserting (7) into (15) yields

$$t^{N\beta - \alpha} \int\limits_{\mathbb{R}^N} f \, \mathrm{d}\boldsymbol{\eta} = M \tag{16}$$

and (16) demands that

$$\alpha = N\beta \tag{17}$$

so, when (9) applies,

$$\alpha = N/(2 - nN), \quad \beta = 1/(2 - nN). \tag{18}$$

(3) *Second-kind self-similarity*
In this final class the similarity exponent (α or a) serves as an eigenvalue in a (nonlinear) eigenvalue problem, i.e., the associated boundary-value problem (which often involves a maximal-decay, or minimal-growth, condition on $f(\boldsymbol{\eta})$ as $\rho \to \infty$) has a non-trivial solution f only when the similarity exponent takes a particular

[3] If the boundary instead took the moving form

$$\omega(\boldsymbol{x}/(t_c - t)^B) = 0,$$

say, for prescribed B and ω, it would be natural to investigate solutions (8) with $b = B$.

value or values (examples in which a countably infinite set of eigenvalues is present are common, but so are cases in which there is a single second-kind solution).

The distinction between solutions of the first and second kind is not as pronounced as the above discussion may imply, as can be illustrated by noting that first-kind solutions might well be constructed from an eigenvalue problem when an applicable conservation law exists but has not been identified. A simple such example follows on making the (rather implausible) supposition that (15) has not been identified from (1): consider radially symmetric solutions in the form (cf. (7))

$$u = t^{-\alpha} f(\rho), \quad \rho = r/t^{\beta} \tag{19}$$

(on setting $r \equiv |\boldsymbol{x}|$, $\rho \equiv |\boldsymbol{\eta}|$) leads to

$$-\alpha\rho^{N-1} f - \beta\rho^{N} \frac{\mathrm{d}f}{\mathrm{d}\rho} = \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{N-1} f^{-n} \frac{\mathrm{d}f}{\mathrm{d}\rho}\right)$$

so that

$$-\beta\rho^{N} f + (N\beta - \alpha)\int_0^{\rho} \rho'^{N-1} f(\rho')\mathrm{d}\rho' = \rho^{N-1} f^{-n} \frac{\mathrm{d}f}{\mathrm{d}\rho}$$

and, on requiring that f be bounded at the origin and tend to zero at infinity, Eq. (17) follows as the condition for a non-negative connection to exist.

3 Symmetry properties

3.1 The parabolic problem

Here we summarise pertinent results from [11] and references therein. For any n, Eq. (1) has obvious translation, rotation and (two) rescaling symmetries, the last of these playing a crucial role in our analysis. In addition, for $n = n_s$ there are additional (conformal) symmetries; these arise from the discrete symmetry

$$u' = r^{N+2}u, \quad \boldsymbol{x}' = \boldsymbol{x}/r^2, \quad r' = 1/r, \; t' = t \tag{20}$$

(where $r' \equiv |\boldsymbol{x}'|$) which maps (1) with $n = n_s$ into itself and when combined with $\boldsymbol{x}$ translations generates the continuous symmetries in question. This is a first property that indicates the exceptional status of $n = n_s$. For $n = 1, \; N = 2$ the symmetry groups extends further, the conformal group being infinite dimensional.

The following property from [11] in some respects generalises the above result to other n. Restricting ourselves to the radially symmetric case, the transformation

$$u' = r^{\frac{N-2}{1-n}} u, \quad r' = 1/r^{\frac{nN-2}{2(1-n)}}, \quad t' = \left(\frac{(nN-2)}{2(1-n)}\right)^2 t \tag{21}$$

maps

$$\frac{\partial u}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} u^{-n} \frac{\partial u}{\partial r}\right) \tag{22}$$

into

$$\frac{\partial u'}{\partial t'} = \frac{1}{r^{N'-1}} \frac{\partial}{\partial r'}\left(r'^{N'-1} u'^{-n} \frac{\partial u'}{\partial r'}\right), \tag{23}$$

where N' need not be an integer, being given by

$$N' = \frac{2(2n + N - 4)}{(nN - 2)}, \tag{24}$$

so that

$$\frac{(nN'-2)}{2(1-n)} = \frac{2(1-n)}{(nN-2)},$$

and

$$\left(N'+2-\frac{4}{n}\right) = -\frac{2(1-n)}{(nN-2)}\left(N+2-\frac{4}{n}\right),$$

the second and third identities indicating symmetry properties (with respect to N and N') of the transformation; moreover, they already suggest that $n = n_s, n = n_c \equiv 2/N$ and $n = 1$ can each be expected to have exceptional status: we note in particular that for $2/N < n < 1$ we have $2/N' < n < 1$ and the subcritical regime $2/N < n < 4/(N+2)$ maps into the supercritical one $4/(N'+2) < n < 1$ (and conversely). The borderline case $n = n_s$ has $N' = N$, corresponding to the invariance under (20).

3.2 The elliptic problem

Adopting the separable ansatz (13) and writing

$$f(\boldsymbol{x}) = \left(\frac{n}{1-n}\right)^{\frac{1}{n}} g(\boldsymbol{x}), \quad p = \frac{1}{1-n} \tag{25}$$

gives the very widely studied elliptic problem

$$\triangle g + g^p = 0. \tag{26}$$

The literature on (26), and on related parabolic equations, is vast and many rigorous results have been established. Those involving the analysis of spike-type solutions are particularly relevant to aspects of what follows; see, for example, [12] for early work and [13] and references therein for an indication of the scope of such results.

The critical Sobolev exponent for (26) is given by $p = p_s$ with

$$p_s = \frac{N+2}{N-2} = \frac{1}{1-n_s}, \tag{27}$$

Equation (26) is of course the Euler–Lagrange equation for

$$\mathcal{L} = \int \left\{\frac{1}{p+1} g^{p+1} - \frac{1}{2}|\nabla g|^2\right\} \mathrm{d}\boldsymbol{x} \tag{28}$$

and the case $p = p_s$ can be identified as exceptional as being the only one in which the continuous symmetries of (26) are all variational symmetries of (28)—in particular, (20) becomes

$$g' = r^{N-2} g, \quad \boldsymbol{x}' = \boldsymbol{x}/r^2, \quad r' = 1/r$$

which leaves (27) invariant. Of most significance here is the rescaling symmetry

$$g' = \lambda^2 g, \quad \boldsymbol{x}' = \lambda^{-(p-1)} \boldsymbol{x} \tag{29}$$

of (26), where λ is here an arbitrary constant. It follows at once that (29) leaves (28) invariant only for $p = p_s$, but pursuing for general p the calculation that leads to a conservation form (that arising from Noether's theorem) when $p = p_s$ gives a result that is of more general applicability here, namely the Pohozaev identity; we refer to [14] (e.g. Chap. 4) and [15, Section 29] for details of Noether's theorem and the Pohozaev identity. We accordingly compute

$$0 = \int g(\Delta g + g^p)\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} g \frac{\partial g}{\partial \nu} \mathrm{d}S + \int (g^{p+1} - |\nabla g|^2)\mathrm{d}\boldsymbol{x},$$

$$\begin{aligned} 0 &= \int (\boldsymbol{x} \cdot \nabla g)(\Delta g + g^p)\mathrm{d}\boldsymbol{x} \\ &= \int_{\partial\Omega} \left\{(\boldsymbol{x} \cdot \nabla g)\frac{\partial g}{\partial \nu} - \frac{1}{2}|\nabla g|^2 \boldsymbol{x} \cdot \boldsymbol{\nu} + \frac{1}{p+1} \boldsymbol{x} \cdot \boldsymbol{\nu} g^{p+1}\right\} \mathrm{d}S \\ &\quad + \int \left(\frac{N-2}{2}|\nabla g|^2 - \frac{N}{p+1} g^{p+1}\right) \mathrm{d}\boldsymbol{x}, \end{aligned}$$

where the volume integrals are taken over either Ω or $\mathbb{R}^N \backslash \Omega$, $\boldsymbol{\nu}$ is the outward unit normal and $\partial g/\partial \nu \equiv \boldsymbol{\nu}.\nabla g$. Hence when $g = 0$ on $\partial\Omega$ we can deduce that

$$\int |\nabla g|^2 \mathrm{d}\boldsymbol{x} = \int g^{p+1} \mathrm{d}\boldsymbol{x}, \tag{30}$$

$$\left(\frac{N-2}{2} - \frac{N}{p+1}\right) \int |\nabla g|^2 \mathrm{d}\boldsymbol{x} = -\frac{1}{2} \int \left(\frac{\partial g}{\partial \nu}\right)^2 (\boldsymbol{x} \cdot \boldsymbol{\nu}) \mathrm{d}S. \tag{31}$$

It follows from (31) that (at least) for a star-shaped domain Ω (whereby we can choose the origin of $\boldsymbol{x}$ such that $\boldsymbol{x} \cdot \boldsymbol{\nu} \geq 0$ on $\partial\Omega$ for the interior problem and $\boldsymbol{x} \cdot \boldsymbol{\nu} \leq 0$ there for the exterior one) we recover the well-known result that $p < p_s$ is a necessary condition for a solution to the interior problem to exist, while we can also infer that $p > p_s$ is needed in the exterior problem. The critical case $p = p_s$ is exceptional in that non-existence follows in *both* cases, this reflecting the presence of the conservation law.

Consistent with the above remarks, we note that for radially symmetric solutions with Ω being the unit ball, (21) maps interior and exterior, as well as subcritical and supercritical, problems into one another.

4 The Cauchy problem

Here we revisit established results (see [9,16] and references therein) for the Cauchy problem, wherein (1) holds in $\mathbb{R}^N$ with

$$\text{at } t = 0 \quad u = u_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^N, \tag{32}$$

with

$$\int_{\mathbb{R}^N} u_0(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = M_0 \tag{33}$$

bounded (a condition that can be relaxed if $n > 2/N$).

Now multiplying (1) by $\Phi(\boldsymbol{x})$ implies that

$$\frac{\partial}{\partial t}(\Phi(\boldsymbol{x})u) = \frac{1}{1-n} \nabla \cdot (\Phi(\boldsymbol{x}) \nabla u^{1-n} - u^{1-n} \nabla \Phi(\boldsymbol{x})) \tag{34}$$

for any $\Phi(\boldsymbol{x})$ that satisfies

$$\Delta \Phi(\boldsymbol{x}) = 0 \tag{35}$$

(cf. [17]). The application of such results to the scaling similarity reductions (7) typically follows on taking

$$\Phi(\boldsymbol{x}) = r^M \Psi(\boldsymbol{x}/r), \quad \tau = \log t, \quad u = t^{-\alpha} f(\boldsymbol{\eta}, \tau), \tag{36}$$

to give

$$\frac{\partial}{\partial \tau}(\rho^M \Psi f) + ((N+M)\beta - \alpha)\rho^M \Psi f = \nabla \cdot \left(\rho^M \Psi f^{-n} \nabla f - \frac{1}{1-n} f^{1-n} \nabla(\rho^M \Psi) + \beta \rho^{M+1} \Psi f \hat{\boldsymbol{\rho}}\right), \tag{37}$$

where the spatial derivatives are now taken with respect to $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\rho}}$ is the unit vector in the ρ direction. Thus setting $\alpha = (N+M)\beta$, so by (9)

$$\alpha = \frac{(N+M)}{2 - n(N+M)}, \quad \beta = \frac{1}{2 - n(N+M)}, \tag{38}$$

Eq. (37) takes a conservation-law form and for appropriate boundary conditions a conserved quantity follows. There are circumstances (cf. [18]) in which it is helpful to write (37) in gradient-flow form. This can readily be achieved (given (38)) in the special cases $\Psi \equiv 1$ (requiring $M = 0$ or $-(N-2)$), in which case

$$\frac{\partial}{\partial \tau}(\rho^M f) = \nabla \cdot \left(\rho^{\frac{1-2n}{1-n}M} f \nabla \left\{ -\frac{\rho^{\frac{n}{1-n}M} f^{-n}}{n} + \frac{\beta(1-n)}{nM + 2(1-n)} \rho^{\frac{n}{1-n}M+2} \right\}\right)$$

and $n = 0$ in which case Φ can be allowed to depend on t as well as $\boldsymbol{x}$, with

$$\frac{\partial \Phi}{\partial t} = -\Delta \Phi,$$

so Φ need not be of the form (36) to be appropriate for scaling self-similarity (cf. [17]), implying (since $\beta = 1/2$)

$$\frac{\partial}{\partial \tau}(\rho^M \Psi f) = \nabla \cdot \left(\rho^M \Psi f \nabla \left\{\log(f/\rho^M \Psi) + \frac{1}{4}\rho^2\right\}\right).$$

In the current case, we simply take $\Phi = 1$ (so $M = 0$, $\Psi = 1$) and integrate over $\mathbb{R}^N$ to recover (15) provided that

$$\lim_{R\to\infty} \int\limits_{r=R} u^{-n} \frac{\partial u}{\partial r} \mathrm{d}S = 0, \tag{39}$$

which is satisfied for (1) provided

$$u = o\left(r^{-\frac{N-2}{1-n}}\right) \quad \text{as} \quad r \to \infty. \tag{40}$$

Recalling that we are restricting attention to $n > 0$ with $N > 2$, the condition (40) is satisfied for $n < n_c$, the critical exponent for the Cauchy problem (see [16] and references therein), in which case $M = M_0$ for all t and (18) holds. Given that the solution is radially symmetric, the well-known Barenblatt solution

$$f^{-n}(\eta) = \frac{n}{2(2-nN)}\left(\ell^2 + \rho^2\right), \quad \rho = r/t^{\frac{1}{2-nN}} \tag{41}$$

follows, wherein the constant ℓ is determined by (15), allowing the leading-order large-time behaviour to be completely characterised. Because first-kind and (more explicitly) second-kind solutions have the status of eigenfunctions,[4] the associated self-similar solutions contain an arbitrary constant (ℓ in the case of (38)); by contrast, zeroth-kind profiles typically contain no such free constants. We note that the requirement for the constraint $n < n_c$, whose necessity we have already noted in the context of (40), is also apparent from both (18) and (41). For $n_c < n < 1$, the far-field behaviour

$$u \sim \left(\frac{(1-n)J(t)}{(N-2)\omega_N}\right)^{\frac{1}{1-n}} r^{-\frac{N-2}{1-n}} \quad \text{as } r \to \infty \tag{42}$$

pertains (contrast (40), which requires faster decay with respect to r and leads to no mass being lost to infinity), where $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the unit hypersphere and $J(t)$ (which must be determined as part of the solution) gives the flux to infinity, with

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -J(t),$$

so that mass is no longer conserved and the relevant similarity solution is one of second-kind, exhibiting finite-time extinction (i.e., of the form (8)), as discussed in [16][5]; in (8) we have $b = 0$ for $n = n_s$, $b < 0$ for $n_c < n < n_s$ (with $b \to -\infty$ as $n \to n_c^+$) and $b > 0$ for $n_s < n < 1$ (with $b \to 1/2$ as $n \to 1^-$). That these second-kind similarity solutions are contracting for $n_s < n < 1$ (but expanding for $n_c < n < n_s$) will be crucial in what follows. No finite-mass solutions exist for $n \geq 1$ (the mass all being lost instantaneously to infinity), while for $n = n_c$ the asymptotics are more involved, the self-similar form that arises not being an exact reduction of (1); again, see [16].

[4] The elliptic boundary-value problem for the similarity solution is scale invariant and has a non-trivial solution only for a particular value (or particular values) of the similarity exponent, which plays the role of an eigenvalue. Because the eigenvalue problem is nonlinear, the arbitrary constant scales the independent variable(s) as well as the dependent one, the non-trivial solution being determined only up to this rescaling.

[5] We conjecture that the extinction behaviour is generically radially symmetric, $f = f(\rho)$; as far as we know, rigorous results in this direction are incomplete.

5 The interior problem

Imposing (12), we observe that the asymptotic behaviour for $0 < n < n_s$ takes the form (13); see [19]. In non-convex domains $f(\boldsymbol{x})$ need not be unique, but we shall not explore such matters here.

For $n_s < n < 1$ the non-existence result on $f(\boldsymbol{x})$ alluded to in Section 2.2 pertains and the asymptotic behaviour is instead the second-kind solution of Section 3, the extinction point $\boldsymbol{x}_c$ (as well as the extinction time t_c) depending on the initial data (as they do for the Cauchy problem). This asymptotic structure is possible only because $b > 0$ for $n_s < n < 1$: the shrinking spike interacts with the boundary $\partial\Omega$ more and more weakly as $t \to t_c^-$ and the asymptotic behaviour thus coincides with that of the Cauchy problem.

This second-kind spike provides the leading-order inner solution as $t \to t_c^-$, the inner scaling being $|\boldsymbol{x} - \boldsymbol{x}_c| = \mathrm{O}((t_c - t)^b)$; this solution has far-field behaviour

$$f(\rho) \sim A\rho^{-\frac{N-2}{1-n}} \quad \text{as } \rho \to +\infty \tag{43}$$

where $A > 0$ also depends on the initial data on the PDE (as indicated above, if $f(\rho)$ satisfies the nonlinear eigenvalue problem then so does $\lambda^{-2/n} f(\rho/\lambda)$ for any constant $\lambda > 0$, $f(\rho)$ being uniquely determined up to this rescaling). The matching condition (43) is equivalent to

$$u^{1-n} \sim A^{1-n}(t_c - t)^{(1-n+(nN-2)b)/n}|\boldsymbol{x} - \boldsymbol{x}_c|^{-(N-2)}. \tag{44}$$

Since $n > n_c, b > 0$, this decays faster than the separable form (13) (and mass is lost from $\partial\Omega$ correspondingly faster) and, as can readily be confirmed by substitution, the leading-order solution in the outer region $\boldsymbol{x} - \boldsymbol{x}_c = \mathrm{O}(1)$, which takes the form

$$u^{1-n} \sim A^{1-n}(N-2)\omega_N(t_c - t)^{(1-n+(nN-2)b)/n} G(\boldsymbol{x} - \boldsymbol{x}_c), \tag{45}$$

satisfies the quasi-steady version of (1). In other words,

$$\Delta(u^{1-n}) = 0 \tag{46}$$

holds at leading order for $\boldsymbol{x} \neq \boldsymbol{x}_c$; more precisely, we have that G is the Green's function:

$$\Delta G = -\delta(\boldsymbol{x} - \boldsymbol{x}_c) \quad \text{for } \boldsymbol{x} \in \Omega, \quad G = 0 \quad \text{for } \boldsymbol{x} \in \partial\Omega \tag{47}$$

(see also [20]). As already noted, the spike position $\boldsymbol{x}_c$ is arbitrary (i.e., depends on the initial data) and the spike can be located arbitrarily close to, but not on, $\partial\Omega$.

Revisiting (45) in the light of our discussion in Section 1 regarding the various classes of similarity solution, we observe that (46) is invariant under three scalings, namely of t (which appears only parametrically in (46)), $\boldsymbol{x} - \boldsymbol{x}_c$ and G. In consequence, in terms of the ansatz (8) the PDE (46) evidently furnishes no constraints on a and b; instead, $b = 0$ follows by requiring invariance of the boundary data on $\partial\Omega$, while the value of a is determined by matching with the inner solution via (44). Thus (45) does not fit naturally within the classification of Section 1, illustrating further the variety of circumstances in which a characterisation into first- and second-kind solutions is insufficient; it should, however, be remarked that it is not in any case usual to undertake such a classification in problems such as this whose asymptotic behaviour subdivides into distinct (inner and outer) regions, the dominant balance in one involving a simplification as drastic as (46).[6] It is instead more appropriate to focus on the region that in effect controls the asymptotic behaviour, this being the (second-kind) inner region in the current example.

Finally, for $n = n_s$ the structure is similar to that for $n_s < n < 1$, comprising a contracting spike that focusses down onto an arbitrary point $\boldsymbol{x}_c$. The main distinction from the supercritical case is that the width of the spike decreases logarithmically, rather than algebraically, in $(t_c - t)$, the solution almost being separable. The relevant structure is described in Appendix 1 (see also [20]), wherein G is given by (47).

[6] An implication of the absence of the time derivative from (46) is that the similarity exponents of the outer solution do not satisfy the constraint (9) arising from the PDE. Problems in which the exponents are not those that would be naively expected on the basis of the PDE itself arise in a number of contexts, notably in so-called Type II blow-up; such problems illustrate the caution that needs to be exercised in attempting to infer asymptotic behaviour from symmetry properties.

6 The exterior problem

The results of [17] and of [21] (for $n < 0$) are relevant to the subcritical case $n < n_c$. In this regime the leading-order outer behaviour is provided by the source solution (41), while the inner region has $r = O(1)$ and in it we have

$$u \sim t^{-\frac{N}{(2-nN)}} \left(\frac{n}{2(2-nN)} \ell^2 \right)^{-\frac{1}{n}} F^p(\boldsymbol{x}) \quad \text{as } t \to +\infty \tag{48}$$

with

$$\Delta F = 0 \quad \text{for } \boldsymbol{x} \in \mathbb{R}^N \backslash \Omega, \quad F = 0 \quad \text{for } \boldsymbol{x} \in \partial\Omega, \quad F \to 1 \quad \text{as } r \to +\infty, \tag{49}$$

where we have matched with (41). Thus, while the asymptotic behaviour (48) is separable in form it is again not a similarity reduction of the full PDE (1). Associated with this asymptotic behaviour are conservation laws obtained by imposing

$$\Phi = 0 \quad \text{for } \boldsymbol{x} \in \partial\Omega \tag{50}$$

on (35). Provided Φ grows sufficiently slowly as $r \to +\infty$ we then have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N \backslash \Omega} \Phi(\boldsymbol{x}) u(\boldsymbol{x}, t) \mathrm{d}\boldsymbol{x} = 0. \tag{51}$$

The slowest growing such solution can be constructed by imposing

$$\Phi \to 1 \quad \text{as } r \to +\infty \tag{52}$$

in addition to (50), this implying that Φ is strictly positive; in the case in which Ω is the unit ball, we then have

$$\Phi = 1 - r^{-(N-2)}.$$

In view of (52), the large-time limit of (51) is (given that the diffusion length grows without bound as $t \to +\infty$) simply conservation of mass, $\Phi \sim 1$, consistent with the source solution describing the associated asymptotics.

A similar asymptotic structure applies for $n_c < n < n_s$ also, though (51) no longer holds for the same reason as before (i.e., the presence of a finite flux to infinity). The outer is the second-kind solution of Section 3 which has $b < 0$ and hence spreads out as $t \to t_c^-$; as when $n > n_c$, this means that the influence of $\partial\Omega$ becomes negligible in the limit and the Cauchy problem accordingly describes the dominant behaviour. The inner region $r = O(1)$ has

$$u \sim (t_c - t)^a f(0) F^p(\boldsymbol{x}), \tag{53}$$

where $f(\rho)$ is the second-kind solution of Section 3 and F again satisfies (49). One significant contrast between (48) and (53) is that ℓ is determined by the value of the integral in (51) associated with the initial data, so that the former is completely specified, whereas the dependence of $f(0)$ in (53) on the initial data cannot be characterised in such a fashion.

In the supercritical case $n_s < n < 1$, the second-kind solution again provides the generic extinction behaviour: because the spike is shrinking, the exterior problem shares the property of the interior one that the asymptotic behaviour is insensitive to the presence of the boundary (as noted above, the subcritical case is also insensitive to the boundary for the exterior problem, though for rather different reasons,[7] but is not for the interior one). The outer behaviour again takes the form (45), with the only change to (47) being that the PDE instead holds for $\boldsymbol{x} \in \mathbb{R}^N \backslash \Omega$. However, a separable solution *does* exist (as can be immediately confirmed for the ball because (21) maps the exterior supercritical problem to the interior subcritical one[8]). This solution is unstable, but plays the following

[7] Namely that, as the solution expands, Ω shrinks in relative size to become negligible.

[8] Applying this transformation to $n < n_c$ generates an interior solution with a singularity at the origin, the interior problem also being subcritical in the sense that $n < 2/N'$

borderline role: consider a one-parameter family of initial data for which, as the parameter varies, the spike moves out to the left (say), with $x_c \to -\infty$, and then reappears on the right, with $x_c \to +\infty$ – the borderline corresponds to this transition from left to right.

The case $n = n_s$ is instructive in regard to the above comments. In this special case the transformation (20) maps the exterior problem to the interior one and, provided the extinction point for the latter is not given by $r'_c \equiv |\boldsymbol{x}'_c| = 0$, the asymptotic structure of the two is identical, comprising a logarithmically contracting spike at the image $\boldsymbol{x}_c = \boldsymbol{x}'_c/r'^2_c$ of the interior extinction point (and its location is accordingly arbitrary). However, in the non-generic case in which $r'_c = 0$, corresponding to $r_c \equiv |\boldsymbol{x}_c|$ being unbounded, the behaviour is rather different. Reading off the relevant results from Appendix 1, if the interior problem has asymptotic solution

$$u' \sim (t_c - t)^{\frac{N+2}{4}}/(\lambda(\tau) + r'^2/4N\lambda(\tau))^{\frac{N+2}{2}},$$

where $\tau = -\log(t_c - t)$ and $\lambda(\tau)$ is given by (A1.12) then for the exterior problem we have

$$u \sim (t_c - t)^{\frac{N+2}{4}}/(\lambda(\tau)r^2 + 1/4N\lambda(\tau))^{\frac{N+2}{2}}. \tag{54}$$

Thus, in this exceptional case (to which the unstable separable solution for $n > n_s$ is analogous) the solution expands, as it does generically for $n < n_s$, but does so only logarithmically: (54) has r scaling with $1/\lambda$, which grows as $(-\log(t_c - t))^{1/(N-2)}$ as $t \to t_c^-$ (see (A1.12)). The inner solution, for $r = O(1)$, is again quasi-steady, cf. (49).

7 The dipole (half-space) problem

Here we impose

$$u = 0 \quad \text{on } x = 0, \tag{55}$$

the $n < 0$ problem having been treated in [22]. Taking $\Phi(\boldsymbol{x}) = x$ in (34)-(35) we have the result

$$\int_{\mathbb{R}^N_+} x u(\boldsymbol{x}, t) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N_+} x u_0(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \tag{56}$$

on the first moment of u, provided that the flux of first moment to infinity is zero, where we use $\mathbb{R}^N_+$ to denote the half-space $x \geq 0$ with the other coordinates spanning $\mathbb{R}^{N-1}$. Inserting (7) into (56) (self-similarity of the first kind) requires $\alpha = (N+1)\beta$ and hence (by (9))

$$\alpha = \frac{N+1}{2 - n(N+1)}, \quad \beta = \frac{1}{2 - n(N+1)}, \tag{57}$$

(these also follow from (38) with $M = 1$) implying that $n < n_d \equiv 2/(N+1)$ is required (so the critical value of n is smaller than it is for the Cauchy problem), a constraint we shall also derive below on other grounds.

The solution then takes the form[9]

$$f(\boldsymbol{\eta}) = f(\eta, \hat{\rho}), \quad \eta = x/t^\beta, \quad \hat{\rho}^2 + \eta^2 = \rho^2,$$

being determined up to a rescaling by the associated boundary-value problem, having (for $0 < n < n_d$)

$$f(\eta, \hat{\rho}) \sim \rho^{-2/n}\Theta(\eta/\rho) \quad \text{as } \rho \to +\infty \tag{58}$$

[9] To clarify the notation in this section, if we denote the Cartesian coordinates by x_i, with $i = 1, 2, \ldots, N$ and $x_1 = x$, and the corresponding similarity variables by $\eta_i = x_i/t^\beta$ with $\eta_1 = \eta$, then

$$r^2 = \sum_{i=1}^N x_i^2, \quad \hat{r}^2 = \sum_{i=2}^N x_i^2, \quad \rho^2 = \sum_{i=1}^N \eta_i^2, \quad \hat{\rho}^2 = \sum_{i=2}^N \eta_i^2.$$

for some function Θ, and being completely specified (i.e., with the rescaling determined) by (56). Thus setting $\hat{r}^2 = r^2 - x^2$, $x = r\cos\theta$, $\hat{r} = r\sin\theta$ and $u = u(r,\theta,t)$ gives

$$\frac{\partial u}{\partial t} = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}u^{-n}\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin^{N-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{N-2}\theta u^{-n}\frac{\partial u}{\partial\theta}\right)$$

and re-expressing Θ as a function of θ gives

$$\frac{1}{\sin^{N-2}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^{N-2}\theta\Theta^{-n}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{2}{n^2}(2-nN)\Theta^{1-n} = \frac{1}{n}\Theta, \tag{59}$$

$$\Theta = 0 \quad \text{on } \theta = \pm\pi/2,$$

(59) being autonomous for $N = 2$. Θ tends to zero as $n \to n_d^-$, but rather than describing this limit we now analyse the borderline case $n = n_d$, in which the large-time behaviour is of an unusual form.

Writing $u = w^{1/(1-n)} = w^{(N+1)(N-1)}$ gives

$$w^{\frac{2}{N-1}}\frac{\partial w}{\partial t} = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin^{N-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{N-2}\theta\frac{\partial w}{\partial\theta}\right)$$

for $n = n_d$ and, adopting the far-field ansatz

$$w \sim t^{\frac{N-1}{2}}r^{-(N-1)}\{\cos\theta W_0(\xi) + \Theta_1(\theta)W_1(\xi)\}$$

as $r \to +\infty$, where $\xi = \log r$, we obtain (assuming slow variation with respect to ξ),

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^{N-2}\theta\frac{\mathrm{d}\Theta_1}{\mathrm{d}\theta}\right) + (N-1)\sin^{N-2}\theta\Theta_1 = \frac{1}{W_1}\left(N\cos\theta\frac{\mathrm{d}W_0}{\mathrm{d}\xi} + \frac{N-1}{2}\cos^{\frac{N+1}{N-1}}\theta W_0^{\frac{N+1}{N-1}}\right)\sin^{N-2}\theta$$

from which we obtain a solvability condition on $\Theta_1(\theta)$ that specifies $W_0(\xi)$, namely

$$N\int_0^{\frac{\pi}{2}}\cos^2\theta\sin^{N-2}\theta\mathrm{d}\theta\frac{\mathrm{d}W_0}{\mathrm{d}\xi} + \frac{N-1}{2}\int_0^{\frac{\pi}{2}}\cos^{\frac{2N}{N-1}}\theta\sin^{N-2}\theta\mathrm{d}\theta W_0^{\frac{N+1}{N-1}} = 0. \tag{60}$$

It follows from (60) that

$$W_0(\xi) \sim \left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{3N-1}{2(N-1)}\right)}{\sqrt{\pi}\Gamma\left(\frac{N(N+1)}{2(N-1)}\right)}\xi\right)^{-\frac{N-1}{2}} \qquad \text{as } \xi \to +\infty,$$

which is easily seen to be (just) consistent with u having bounded first moment for $n = n_d$, $N > 2$. Thus we have

$$u \propto t^{\frac{N+1}{2}}r^{-(N+1)}\log^{-(N+1)/2} r \quad \text{as } r \to +\infty. \tag{61}$$

For the corresponding case $n = n_c$ for the Cauchy problem, it is shown in [16] that an expression corresponding to (61) provides the outer solution in the $t \to +\infty$ asymptotic structure and a similar structure applies here also: adopting the ansatz

$$u \sim t^{-\alpha}\mathrm{e}^{-\mu t^\gamma}f(r/t^\beta\mathrm{e}^{\nu t^\gamma}, \theta) \quad \text{as } t \to +\infty \tag{62}$$

for the inner solution, this being a modulated version of the true similarity reduction

$$u = \mathrm{e}^{-(N+1)\nu t}f(\boldsymbol{x}/\mathrm{e}^{\nu t}) \quad \text{for } n = n_d,$$

one finds on substituting in (1) that

$$\mu = (N+1)\nu, \quad \gamma = 1 + \frac{2}{N+1}\alpha - 2\beta. \tag{63}$$

The required far-field behaviour of (62) is of the (quasi-steady) form (58), so the associated first moment would grow as $\log r$ as $r \to +\infty$ (the presence of the outer region, whereby (61) holds, being necessary to obtain the desired bounded first moment); since (62) has $\log r$ behaving as t^{γ}, requiring the 'first moment' associated with (62) to be independent of t requires

$$\mu = (N+1)\nu, \quad \gamma = \alpha - (N+1)\beta$$

which, together with (63), implies

$$\gamma = \frac{N+1}{N-1}, \quad \alpha = (N+1)\beta + \frac{N+1}{N-1}. \tag{64}$$

Thus γ is determined, but ν and β are not: their calculation requires higher-order matching and is not straightforward (cf. that in [16] for the much simpler (radial symmetric) Cauchy-problem critical case, $n = n_c$). Notwithstanding, it is worth emphasising the somewhat unusual self-similar form (62), wherein the dominant time dependence can be characterised by the value of γ in (64).

For $n_d < n < n_s$ there is a flux $I(t)$ of first moment to infinity,

$$u^{1-n} \sim 2(1-n)I(t)x/\omega_N r^N \quad \text{as } r \to +\infty, \tag{65}$$

where $I(t)$ is to be determined as part of the solution, so that (56) no longer holds; rather, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}_+^N} xu\mathrm{d}\boldsymbol{x} = -I(t).$$

In this regime we have finite-time extinction, this being associated with self-similarity of the second kind as in (8), with $a \to +\infty$, $b \to -\infty$ as $n \to n_d^+$ and $a \to (1/n)^+$, $b \to 0^-$ as $n \to n_s^-$, the latter limit being addressed in Appendix 2: $b < 0$ holds throughout this regime, associated with the (non-radially symmetric) solution being of the expanding class. We note that, while full radial symmetry cannot be attained due to the boundary data, we conjecture that the asymptotic behaviour depends only on η and $\hat{\rho}$, representing (for generic initial data) a considerable gain in symmetry.

Finally, for $n_s < n < 1$ we return to the second-kind solution of Section 3 (the shrinking spike), the outer behaviour again being given by the relevant Green's function; here $b > 0$ holds and the limit behaviour is radially symmetric. For $n = n_s$ we similarly have a (logarithmically) shrinking spike, as can readily be inferred either directly by the analysis of Appendix 1 or by first mapping the domain to the interior of a circle (say by replacing x by $x+1$ in (20) to map the half space to the interior of $(x'-1/2)^2 + \hat{r}'^2 = 1/4$, where $\hat{r}'^2 = r'^2 - x'^2$) and appealing to the results of Section 4.

8 Discussion

An issue that warrants revisiting is that of spike location. Spikes arise both in the limit $n \to n_s^-$ and in the limit $t \to t_c^-$ with $n_s \le n < 1$. In the former case, their asymptotic location can be determined, being (under appropriate uniqueness assumptions on (26)) independent of the initial data (see [23] for the elliptic problem corresponding to the separable solution and Appendix 2 for the evolution in the dipole case, that limit being treated in detail there in part to illustrate how spike dynamics can be analysed). In the latter case, however, their location does depend on the initial data; since it is not clear a priori that higher-order terms in the expansion do not in fact specify where the spike has to be located, it is worth making the following remarks about a special case, namely $n = n_s$ (right on the borderline, where it would be thought most likely that the property of the $n \to n_s^-$ limit that the spike location is independent of the initial data might carry over into $n \ge n_s$) in the simplest geometry, namely the unit ball (for which on symmetry grounds the spike might be expected to be located at the centre unless its position is arbitrary). Taking the unit ball to be

$$(x - x_0)^2 + \hat{r}^2 = 1 \tag{66}$$

and applying the transformation (20) with $|x_0| > 1$ (so the interior is mapped to the interior), we obtain the sphere

$$\left(x' - \frac{x_0}{x_0^2 - 1}\right)^2 + \hat{r}'^2 = \frac{1}{(x_0^2 - 1)^2} \tag{67}$$

of radius $1/(x_0^2 - 1)$. Now the centre $(x_0, 0)$ of (66) maps under (20) to $(x', \hat{r}') = (1/x_0, 0)$, which does not coincide with the centre $(x_0/(x_0^2 - 1), 0)$ of (67). Thus the centre of the ball can have no special status for $n = n_s$, in keeping with the assertions above.

We have paid particular attention in the foregoing to the special case $n = n_s$ (which has a geometrical interpretation in terms of Yamabe flow), both because its rich symmetry structure makes it in some respects more tractable than the other cases and because it plays a crucial role as a critical case, and in this regard its analysis is in other respects more challenging (due in particular to the appearance of logarithmic terms in the asymptotics). Thus, the separate analyses of the regimes $n_c < n < n_s$ and $n_s < n < 1$, in which the asymptotics are more readily accessible, guide that of the more delicate bordercase $n = n_s$ that is sandwiched between them; conversely, the exceptional properties of $n = n_s$ lead to certain insights that carry over to the generic regimes. The separable (or near-separable) solution is ubiquitous in our analyses of $n = n_s$: that it arises in the asymptotic treatment of each of the initial boundary-value problems above is associated with the exceptional properties of (26) when $p = p_s$, these being context (i.e., boundary-condition) independent. Thus for $n = n_s$ the separable solution can be expected a priori to play a crucial role; by contrast, the second-kind solutions that in some respects provide continuations of this separable solution to other n often depend significantly on the boundary data (as exemplified by the Cauchy and dipole problems with $n < n_s$). The conformal invariance (20) of $n = n_s$ allows the similarity exponents for the Cauchy problem to be identified beforehand, a characteristic shared with first-kind solutions (albeit resulting here from the presence of an additional symmetry rather than, as is conventional, of a conserved quantity). It is worth revisiting the other exceptional properties of the case $n = n_s$, which include:

(i) it has additional (conformal) symmetries, the symmetries of the elliptic problem for the separable solution all being variational;
(ii) it provides the borderline between spreading and shrinking second-kind solutions;
(iii) in keeping with (ii), the associated second-kind solution is separable and is thus (a) neutrally stable with respect to spatial translations and (b) borderline in terms of whether it interacts more or less strongly with a fixed boundary as the extinction time is approached, both of these properties illustrating its role as a bifurcation point;
(iv) it is unique for $n < 1$ in that neither the interior nor the exterior problem has a separable solution—this non-existence underpins the amount of analytical progress that is possible in describing the transition: the non-existence of a solution for $n = n_s$ implies that the limit is a singular one, allowing analytical treatment local to the bifurcation point despite the inapplicability of conventional linear or weakly nonlinear methods there.

The case $n = n_s$ thus has special status from the points of view of differential geometry, PDE analysis, symmetry methods and studies of intermediate asymptotics, for distinct (but related) reasons. In the light of (iii)(a), the unstable separable solution to the exterior problem for $n \in (n_s, 1)$ can be interpreted as a profile that attempts to centre itself on Ω and is accordingly unstable to a translational mode that will drive it to one side of Ω or the other, resulting in a contracting spike within $\mathbb{R}^N \backslash \Omega$. As n increases, secondary bifurcations are expected to occur, making the separable solution more and more unstable. A related manifestation of borderline role of $n = n_s$ is that noted above: possible spike locations in the limit $n \to n_s^-$ are independent of the initial data, whereas for $n_s < n < 1$ the spike is unstable to translations, making its location initial-condition dependent.

In order to set up our concluding remarks, we next seek to clarify the extent to which the above formal results are new. The Cauchy problem has been the subject of a great deal of analysis; the role of the associated second-kind solution had not, however, previously been analysed in the exterior or half-space problems (and the critical case $n = n_s$ had previously been analysed, [20], only for the radially symmetric interior problem). Fast diffusion ($n > 0$) seems previously to be largely unexplored for both the exterior and half-space problems, with the non-radial second-kind solutions to the latter being of particular interest. A number of natural generalisations suggest themselves, the analysis of multiply connected domains being particularly instructive. Another would be to explore the possible role of (unstable) solutions that, in the spike-like limit, correspond to objects of dimension greater than zero (co-dimension N, corresponding to the supercritical second-kind solutions to the Cauchy problem) but less than $N-1$ (co-dimension 1, corresponding to the unstable separable solutions that exist for the supercritical exterior problem, for which the limit $n \to 1^-$ pertains in describing a localised structure); given the values of the various critical exponents, such scenarios seem plausible only for $N > 3$.

We conclude by briefly re-expressing some of our results in the form of conjectures suitable for rigorous investigation.

(A) In the borderline case $n = n_s$, logarithmic modifications of the separable solution are conjectured to describe the intermediate-asymptotic behaviour of all the initial boundary-value problems we have considered (cf. Appendix 1). Rigorous confirmation of this ubiquity and of our characterisation of the logarithmic dependencies upon $t_c - t$, including in the non-radially symmetric setting, would be valuable.

(B) For $n_s < n < 1$, the contracting second-kind solution is similarly ubiquitous, and our analysis assumes it necessarily to be radially symmetric: rigorous confirmation of this would be valuable.[10] The contrary result (that the radially symmetric solution is *not* generic) would have significant implications for each of the problems studied above.

(C) The boundary data for the half-space problem proclude a radially symmetric form for an expanding asymptotic solution. Thus the second-kind problem for $n_d < n < n_s$ in Section 7 requires consideration of a fully two-dimensional (in ρ and Θ or η and $\hat{\rho}$) elliptic problem, making it rather unusual. Analysis of this elliptic equation, and of its role is the intermediate-asymptotic description of the parabolic problem, would be instructive.

(D) The secondary bifurcation properties of the exterior separable solution warrant investigation (both formal and rigorous).

(E) More generally, rigorous results are in very short supply in the supercritical case, other than for the Cauchy problem.

We hope that these topics will be taken up in due course.

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Appendix 1: Asymptotic results for the critical Sobolev case

The results given here are relevant to each of the initial-boundary-value problems discussed above. Setting

$$u = (t_c - t)^{\frac{1}{n}} f(\boldsymbol{x}, \tau), \quad \tau = -\log(t_c - t)$$

in (1) yields

$$\frac{\partial f}{\partial \tau} - \frac{1}{n} f = \nabla \cdot (f^{-n} \nabla f). \tag{A1.1}$$

For the purposes of the current calculation it is expedient then to set $f = \phi^{-2/n}$ to give the cubically nonlinear equation

$$\frac{\partial \phi}{\partial \tau} + \frac{1}{2}\phi = \phi^2 \Delta\phi - \left(\frac{2}{n} - 1\right) \phi |\nabla\phi|^2, \tag{A1.2}$$

so for $n = n_s$

[10] It is worth noting that in other contexts singular finite-time behaviour (such as blow-up in (A2.1) and some of its relatives) often need not be radially symmetric. Intuition into this can be gleaned by noting that the diffusion length for (A2.1) grows as $t^{1/2}$, so that it is problematic for u to redistribute itself to a locally radially symmetric form at finite t; by contrast, extinction in (1) has the diffusivity $D(u) = u^{-n} \to +\infty$ as $u \to 0$ with $n > 0$, so that u may indeed have the capacity to attain radial symmetry at the extinction point at a finite extinction time.

$$\phi\Delta\phi - \frac{N}{2}|\nabla\phi|^2 - \frac{1}{2} = \frac{\partial\phi}{\partial\tau}/\phi, \tag{A1.3}$$

having a family of steady solutions

$$\phi = \lambda + r^2/4N\lambda \tag{A1.4}$$

in which λ is an arbitrary constant (it is the simplicity of (A1.4) that motivates the introduction of ϕ) and where in this appendix only we define $r \equiv |\boldsymbol{x} - \boldsymbol{x}_c|$. Now allowing λ to depend weakly on τ, we set $r = \lambda(\tau)R$ and introduce the expansion

$$\phi \sim \lambda(\tau)\phi_0(R) + \dot{\lambda}(\tau)\phi_1(R) \tag{A1.5}$$

where $^{\cdot} \equiv \mathrm{d}/\mathrm{d}\tau$, $\phi_0(R) = 1 + R^2/4N$ (se (A1.4)) and we emphasise that ϕ_1 is also taken to be radially symmetric. Thus

$$\left(1 + \frac{R^2}{4N}\right)\frac{1}{R^{N-1}}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^{N-1}\frac{\mathrm{d}\phi_1}{\mathrm{d}R}\right) - \frac{R}{2}\frac{\mathrm{d}\phi_1}{\mathrm{d}R} + \frac{1}{2}\phi_1 = \frac{1 - \dfrac{R^2}{4N}}{1 + \dfrac{R^2}{4N}} \tag{A1.6}$$

Because (A1.4) is a solution for arbitrary λ, differentiating it with respect to λ identifies $\phi_1 = 1 - R^2/4N$ as a solution of the homogeneous version of (A1.6) and we pursue the method of reduction of order by setting $\phi_1 = (1 - R^2/4N)\omega$ to give

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}R^2} + \left(\frac{N-1}{R} - \frac{R}{N}\frac{1}{(1 - R^2/4N)} - \frac{R}{2}\frac{1}{(1 + R^2/4N)}\right)\frac{\mathrm{d}\omega}{\mathrm{d}R} = \frac{1}{(1 + R^2/4N)^2},$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{R^{N-1}(1 - R^2/4N)^2}{(1 + R^2/4N)^N}\frac{\mathrm{d}\omega}{\mathrm{d}R}\right) = \frac{R^{N-1}(1 - R^2/4N)^2}{(1 + R^2/4N)^{N+2}}$$

and hence

$$\frac{\mathrm{d}\omega}{\mathrm{d}R} = \frac{1}{2}(2\sqrt{N})^N\frac{(1 + R^2/4N)^N}{R^{N-1}(1 - R^2/4N)^2}\int_0^{R^2/4N}\frac{\sigma^{\frac{N-2}{2}}(1-\sigma)^2}{(1+\sigma)^{N+2}}\mathrm{d}\sigma. \tag{A1.7}$$

For matching purposes we require the behaviour of ω as $R \to +\infty$ and, using standard properties of the Euler beta function and observing that the apparent singularity at $R = 2\sqrt{N}$ is not in fact troublesome, we obtain from (A1.7) that

$$\frac{\mathrm{d}\omega}{\mathrm{d}R} \sim \frac{\Gamma^2(N/2)}{2^{N-3}N^{\frac{N}{2}-3}(N+1)!}R^{N-3} \quad \text{as } R \to +\infty \tag{A1.8}$$

and hence

$$\phi_1 \sim -\frac{\mu_N}{2N(N-2)}R^N \quad \text{as } R \to +\infty, \;\; \mu_N \equiv \frac{\Gamma^2(N/2)}{2^{N-2}N^{\frac{N}{2}-3}(N+1)!}. \tag{A1.9}$$

Expressing the relevant matching condition in the appropriate variables (using (A1.5)), we have as $R \to +\infty$ that

$$u^{1-n} \sim (t_c - t)^{\frac{N-2}{4}}(4N)^{\frac{N-2}{2}}\lambda^{\frac{N-2}{2}}\left(\frac{1}{r^{N-2}} + \mu_N\frac{\dot{\lambda}}{\lambda^{N-1}}\right). \tag{A1.10}$$

Thus if the relevant Green's function (i.e., that describing the outer behaviour) has

$$G \sim \frac{1}{(N-2)\omega_N}\left(\frac{1}{r^{N-2}} - \frac{1}{L^{N-2}}\right) \quad \text{as } r \to 0, \tag{A1.11}$$

where the constant L depends only on Ω and $\boldsymbol{x}_c$, it follows that

$$\lambda(\tau) \sim L((N-2)\tau/\mu_N)^{-\frac{1}{N-2}} \quad \text{as } \tau \to +\infty. \tag{A1.12}$$

While (A1.12) is the main result of this appendix, it is worth also recording a transformation of the radially symmetric problem (22) with $n = n_s$ that provides more insight into why the analytic progress above is possible. Setting

$$u = r^{-\frac{N+2}{2}} w^{\frac{N+2}{N-2}}, \quad \xi = \log r \tag{A1.13}$$

we obtain

$$w^{\frac{4}{N-2}} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} - \left(\frac{N-2}{2}\right)^2 w \tag{A1.14}$$

(the absence of a $\partial w/\partial \xi$ term from the right-hand side is specific to the choice $n = n_s$). The separation-of-variables reduction

$$w(\xi, t) = (t_c - t)^{\frac{N-2}{4}} \theta(\xi)$$

then leads to

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} = \left(\frac{N-2}{2}\right)^2 \theta - \frac{N-2}{4} \theta^{\frac{N+2}{N-2}}$$

which is a particularly convenient form of the separable ODE and is readily solved in terms of quadratures. Underlying these results is that the scaling invariant of (26) is, for $n = n_s$, a variational symmetry for (28).

Appendix 2: The 'spike' limit of the dipole problem

A2.1 Preamble

In this appendix we consider the behaviour of the dipole problem as $n \to n_s^-$, for two reasons. Firstly, this is one of the simplest contexts in which spike dynamics can be addressed, it being important to emphasise that for (1) the spikes that arise in this limit are stable, in contrast to those of

$$\frac{\partial w}{\partial t} = \Delta w + w^p, \tag{A2.1}$$

say,[11] the steady-state solutions to which satisfy the same elliptic problem as the separable solutions to (1). Secondly, the second-kind solutions for $n_d < n < n_s$ are non-radial in form, unlike those of the Cauchy problem, and the limit explored below is one that allows such non-radial solutions to be characterised explicitly for the first time.

In part to make the analysis resemble closely that of (A2.1), after setting

$$u = (t_c - t)^{\frac{1}{n}} f(\boldsymbol{x}, \tau), \tau = -\log(t_c - t), \tag{A2.2}$$

we define $g(\boldsymbol{x}, \tau)$ according to (25) to give

$$(p-1)g^{p-1} \frac{\partial g}{\partial \tau} = \Delta g + g^p. \tag{A2.3}$$

Non-existence of a separable solution for u can be inferred from the results of Section 2, with the right-hand side of (31) being zero.

[11] More precisely: the unstable mode of the separable solution to (1) simply perturbs the extinction time, maintaining the separable form; that of the steady state of (A2.1) drives the solution into an entirely different class (finite-time blow-up on one side and infinite-time (linearly diffusive) extinction on the other).

A2.2 $N = 2$, $p = 1/\varepsilon$

Here only in this paper we consider the two-dimensional case; for $N = 2$ we have $n_d = 2/3$, $n_s = 1$, the dipole problem being the only one amongst those we study in which there is a gap for $N = 2$ between the relevant critical exponent and n_s. The inner scalings for $0 < \varepsilon \ll 1$ are (cf. [24], from which a number of the results needed here are taken, though our notation will differ somewhat)

$$g(\boldsymbol{x}, \tau) = \lambda(\tau)(1 + \varepsilon h(\boldsymbol{X}, \tau)), \quad \boldsymbol{X} = \varepsilon^{-\frac{1}{2}} \lambda^{\frac{1-\varepsilon}{2\varepsilon}}(\tau)(\boldsymbol{x} - \boldsymbol{x}_c(\tau)), \; R \equiv |\boldsymbol{X}|, \tag{A2.4}$$

where $\boldsymbol{x}_c$ (the point at which $g(\boldsymbol{x}, \tau)$ is maximal for given τ) and λ (the maximum value of g, i.e., $\lambda(\tau) \equiv g(\boldsymbol{x}_c, \tau)$ so that $h = 0$ at $R = 0$) remain to be determined and we can without loss of generality set $\boldsymbol{x}_c = (x_c, 0)$. Thus

$$(1 + \varepsilon h)^{\frac{1-\varepsilon}{\varepsilon}} \left(\varepsilon\lambda \frac{\partial h}{\partial \tau} + \dot{\lambda}(1 + \varepsilon h) + \frac{1 - \varepsilon}{2} \dot{\lambda} R \frac{\partial h}{\partial R} - \varepsilon^{\frac{1}{2}} \lambda^{\frac{1+\varepsilon}{2\varepsilon}} \dot{x}_c \frac{\partial h}{\partial X} \right) = \frac{\varepsilon\lambda}{1 - \varepsilon} (\Delta h + (1 + \varepsilon h)^{\frac{1}{\varepsilon}}). \tag{A2.5}$$

We shall find that $\lambda > 1$, so that some of the terms in (A2.5) are exponentially smaller than others.

We need only to construct the leading-order (radial) term in the solution to (A2.5), together with the first non-radial one. The former is given by (see [24] and references therein)

$$h_0 = -2 \log(1 + R^2/8) \tag{A2.6}$$

because the left-hand side of (A2.5) is negligible for these purposes, the solution being slowly varying with respect to τ, i.e., 'near-separable'. The latter is driven by the final term on the left-hand side of (A2.5); introducing a contribution

$$\varepsilon^{-\frac{1}{2}} \lambda^{\frac{1-\varepsilon}{2\varepsilon}} \dot{x}_c X H(R) \tag{A2.7}$$

into the expansion for h gives

$$\frac{\mathrm{d}^2 H}{\mathrm{d}R^2} + \frac{3}{R} \frac{\mathrm{d}H}{\mathrm{d}R} + \mathrm{e}^{h_0} H = -\frac{1}{R} \mathrm{e}^{h_0} \frac{\mathrm{d}h_0}{\mathrm{d}R},$$

i.e.,

$$\frac{\mathrm{d}^2 H}{\mathrm{d}R^2} + \frac{3}{R} \frac{\mathrm{d}H}{\mathrm{d}R} + \frac{1}{(1 + R^2/8)^2} H = \frac{1}{2(1 + R^2/8)^3}. \tag{A2.8}$$

Now it follows by the x-translation invariance of (1) that

$$\frac{1}{X} \frac{\partial h_0}{\partial X} = -\frac{1}{2(1 + R^2/8)}$$

must be part of the complementary function of the linear ODE (A2.8), so we apply the method of reduction of order and set

$$H(R) = \frac{1}{1 + R^2/8} \Lambda(R)$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}R} \left(\frac{R^3}{(1 + R^2/8)^2} \frac{\mathrm{d}\Lambda}{\mathrm{d}R} \right) = \frac{R^3}{2(1 + R^2/8)^4}$$

so that (imposing the required regularity at $R = 0$)

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}R} = \frac{R(3 + R^2/8)}{24(1 + R^2/8)}.$$

Hence

$$\Lambda(R) \sim \frac{R^2}{48}, \quad H(R) \sim \frac{1}{6} \text{ as } R \to +\infty. \tag{A2.9}$$

We are now in a position to match with the outer region $|\boldsymbol{x}-\boldsymbol{x}_c| = O(1)$ wherein the solution is simply a multiple of the Green's function

$$G(\boldsymbol{x};\boldsymbol{x}_c) = -\frac{1}{4\pi}\log\left((x-x_c)^2+y^2\right)+\frac{1}{4\pi}\log\left((x+x_c)^2+y^2\right), \tag{A2.10}$$

namely

$$g \sim 8\pi\varepsilon\lambda G, \tag{A2.11}$$

where the coefficient has been obtained by matching the first term in (A2.10) with the logarithmic dependence of the far-field of (A2.7); indeed, rewriting (A2.10)–(A2.11) in inner variables gives

$$g \sim \lambda\left(-4\varepsilon\log R+2(1-\varepsilon)\log\lambda+2\varepsilon\log(1/\varepsilon)+4\varepsilon\log(2x_c)+\frac{2\varepsilon^{\frac{3}{2}}}{x_c}\lambda^{-\frac{1-\varepsilon}{2\varepsilon}}X\right); \tag{A2.12}$$

matching all terms but the last of (A2.12) with (A2.4), (A2.6) requires

$$2(1-\varepsilon)\log\lambda+\varepsilon\log(1/\varepsilon)+4\varepsilon\log(2x_c)\sim 1+6\varepsilon\log 2,$$

i.e.,

$$\lambda\sim \mathrm{e}^{\frac{1}{2}},\quad \log\lambda=\frac{1}{2}-\varepsilon\log(1/\varepsilon)+\varepsilon\left(\frac{1}{2}+\log 2-2\log x_c\right)+o(\varepsilon). \tag{A2.13}$$

Moreover, matching (A2.4), (A2.7), (A2.9) with the final term in (A2.12) requires

$$\frac{1}{6}\varepsilon^{\frac{1}{2}}\lambda^{\frac{1-\varepsilon}{2\varepsilon}}\dot{x}_c\sim\frac{2\varepsilon^{\frac{3}{2}}\lambda^{-\frac{1-\varepsilon}{2\varepsilon}}}{x_c}.$$

Thus (A2.7) is exponentially small in ε when $x_c = O(1)$ and, using (A2.13),

$$\dot{x}_c\sim 6\mathrm{e}^{-\frac{1}{2\varepsilon}}x_c. \tag{A2.14}$$

The variation of x_c and λ with τ is thus in fact exponentially slow and, tracking back through the various transformations, we find in (8) that

$$a\sim\frac{1}{n}+12\mathrm{e}^{-\frac{1}{2\varepsilon}},\quad b\sim -6\mathrm{e}^{-\frac{1}{2\varepsilon}}\ \text{as}\ \varepsilon\to 0. \tag{A2.15}$$

A2.3 $N > 2$, $p = p_s - \varepsilon$

The appropriate inner scalings are now (again cf. [24])

$$g(\boldsymbol{x},\tau)=\lambda(\tau;\varepsilon)h(\boldsymbol{X},\tau),\quad \boldsymbol{X}=\lambda^{\frac{2}{N-2}-\frac{\varepsilon}{2}}(\boldsymbol{x}-\boldsymbol{x}_c(\tau)),$$

where $\boldsymbol{x}=(x,\hat{r})$, $\boldsymbol{x}_c=(x_c,0)$ and we here make explicit the dependence of λ upon ε, since λ will prove to be large with respect to ε (and will, like x_c, be slowly varying in τ).

Two radially symmetric terms are needed, namely

$$h\sim h_0(R)+\varepsilon h_1(R),$$

where

$$\Delta h_0+h_0^{\frac{N+2}{N-2}}=0,$$

so that

$$h_0=(1+R^2/N(N-2))^{-\frac{N-2}{2}} \tag{A2.16}$$

and the computation in [24] gives

$$h_1\sim-\frac{(N-2)\Gamma^2(N/2)}{4\Gamma(N)}\quad\text{as } R\to+\infty; \tag{A2.17}$$

self-consistency of the above expansion requires that $\dot{\lambda}/\lambda = o(\varepsilon)$ and we shall find in due course that $\dot{\lambda}/\lambda = O(\varepsilon^{N/(N-2)})$, consistent with this constraint. The first non-radial contribution to h in the inner expansion is of the form

$$\lambda^{\frac{2}{N-2}-\frac{\varepsilon}{2}}\dot{x}_c X H(R), \tag{A2.18}$$

with

$$\frac{\mathrm{d}^2 H}{\mathrm{d}R^2} + \frac{N+1}{R}\frac{\mathrm{d}H}{\mathrm{d}R} + \frac{N+2}{N-2}\frac{H}{(1+R^2/N(N-2))^2} = \frac{4}{N(N-2)}\frac{1}{(1+R^2/N(N-2))^{\frac{N+4}{2}}}. \tag{A2.19}$$

An x-translation-invariance argument in this case identifies

$$\frac{1}{X}\frac{\partial h_0}{\partial X} = -\frac{1}{N(1+R^2/N(N-2))^{\frac{N}{2}}}$$

as part of the complementary function of (A2.19), so we set

$$H(R) = \frac{1}{(1+R^2/N(N-2))^{\frac{N}{2}}}\Lambda(R)$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{R^{N+1}}{(1+R^2/N(N-2))^N}\frac{\mathrm{d}\Lambda}{\mathrm{d}R}\right) = \frac{4}{N(N-2)}\frac{R^{N+1}}{(1+R^2/N(N-2))^{N+2}}$$

from which it readily follows that

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}R} \sim \frac{N^2\Gamma^2(N/2)}{2(N(N-2))^{N/2}(N+1)!}R^{N-1} \quad \text{as } R \to +\infty$$

and hence that

$$H \sim \frac{N\Gamma^2(N/2)}{2(N+1)!} \quad \text{as } R \to +\infty. \tag{A2.20}$$

The leading-order outer solution in this case takes the form

$$g \sim \frac{(N(N-2))^{\frac{N-2}{2}}}{\lambda^{1-\varepsilon\frac{(N-2)}{2}}}\left(\frac{1}{((x-x_c)^2+\hat{r}^2)^{\frac{N-2}{2}}} - \frac{1}{((x+x_c)^2+\hat{r}^2)^{\frac{N-2}{2}}}\right), \tag{A2.21}$$

where matching of the singular part of this Green's function with the large-R behaviour of (A2.16) has already been accomplished. Matching the constant term in the expansion of (A2.21) about $(x_c, 0)$ with (A2.17) requires $\lambda = O(\varepsilon^{-1/2}) \to +\infty$ as $\varepsilon \to 0^+$ and, more precisely, that

$$\lambda^2 \sim \frac{4\Gamma(N)(N(N-2))^{\frac{N-2}{2}}}{(N-2)\Gamma^2(N/2)(2x_c)^{N-2}}\frac{1}{\varepsilon}, \tag{A2.22}$$

while appropriately matching the x-derivative of (A2.21) at $\boldsymbol{x} = (x_c, 0)$ with (A2.18), (A2.20) requires

$$\frac{N\Gamma^2(N/2)}{2(N+1)!}\lambda^{\frac{N}{N-2}}\dot{x}_c \sim \frac{1}{\lambda^{\frac{N}{N-2}}}N^{\frac{N-2}{2}}(N-2)^{\frac{N}{2}}\frac{1}{(2x_c)^{N-1}}$$

so that the spike dynamics are in this case governed by

$$\dot{x}_c \sim \varepsilon^{\frac{N}{N-2}}\frac{(N+1)(N-2)^{\frac{N}{N-2}}\Gamma^{\frac{4}{N-2}}(N/2)}{2^{\frac{4}{N-2}}N\Gamma^{\frac{2}{N-2}}(N)}x_c, \tag{A2.23}$$

where we have made use of (A2.22). Hence in (8) we have

$$a \sim \frac{1}{n} - \frac{N+2}{2}b, \quad b \sim -\varepsilon^{\frac{N}{N-2}}\frac{(N+1)(N-2)^{\frac{N}{N-2}}\Gamma^{\frac{4}{N-2}}(N/2)}{2^{\frac{4}{N-2}}\ N\Gamma^{\frac{2}{N-2}}(N)} \text{ as } \varepsilon \to 0,$$

using both (A2.22) and (A2.23); b is again small and negative, associated with a (near-separable) expanding similarity solution.

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